ON SIR ISAAC NEWTON'S FIRST SOLUTION OF THE PROBLEM FOR FINDING THE RELATION BETWEEN RESISTANCE AND GRAVITY, THAT A BODY MAY BE MADE TO DESCRIBE A GIVEN CURVE; AND THE SOURCE OF ERROR IN THAT SOLUTION POINTED OUT.

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IT is well known, that Newton's investigation of this problem, as published in the first edition of the Principia,* is erroneous. In the subsequent editions, the illustrious author has given an accurate solution, by an entirely different method, and without adverting to the former solution. John Bernouilli seems to have been the first who pointed out the erroneous conclusion in the first edition.† Nich. Bernouilli imagined he had discovered the source of error in the Newtonian solution. His opinion seems to have been generally acquiesced in till lately, when the celebrated Lagrange, in his ingenious work, entitled "Theorie des Fonctions analytiques," remarked, that Newton's solution is accurate in the part in which N. Bernouilli had thought it erroneous. Indeed had the error been such as was pointed out by N. Bernouilli,

* Lib. II. Prob. 3.

† Mem. Acad. Scien. 1711. & Tom. I. opera Bernouilli.

Bernouilli, nothing would have been easier than to have made the necessary correction; and it would not have been requisite for Newton to have invented that solution which he has given in the second and third editions of the Principia, and which is much more intricate than the former would have been when so corrected.

Lagrange, in the above-cited work, has given Newton's solution somewhat simplified, deducing the same conclusion as Newton, but does not attempt to point out the precise error. He then gives a second solution, which he considers as proceeding upon the same principles as the solution of Newton, and by which he obtains the same result. He points out the precise error of his second solution, and concludes, that the error in the solution of Newton is of the same kind.

Now it may be remarked, that the process of Newton is entirely different in all its steps from that of Lagrange. Therefore they have no common error: but as they give the same result, the error in each must admit of being traced to a common source. What that common source is, M. Lagrange has not shewn. And it still appeared an object of some importance, to enquire into the precise error of the Newtonian solution. The conclusion, deduced by that solution, is confessedly wrong; and, therefore, error must exist in some of its steps, and be assignable without reference to any other solution. To point out that error, is the principal object of this paper. It will be found to have originated from an erroneous application of the method of prime and ultimate ratios. It will also appear, that, had not this error occurred,

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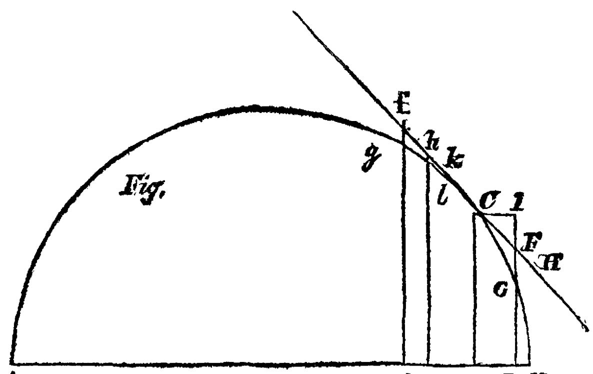
the process would have been very intricate, and it would have been difficult to have obtained the conclusion required. This makes it highly probable, that Newton himself discovered the precise error; and, on account of the subsequent difficulty, abandoned this mode of solution for that which he afterwards gave.

The Newtonian solution is here given in the words of the author, as well as two of the solutions of Lagrange. This seemed necessary for making intelligible the remarks on these solutions. This paper concludes with a solution deduced entirely from the method of limits and series.



" Prob.* Tendat uniformis vis gravitatis directe ad pla-
" num horizontis, sitque resistentia ut medii densitas et
" quadratum velocitatis conjunctim: requiritur tum medii
" densitas in locis singulis, quæ faciat ut corpus in data
" quavis linea curva moveatur, tum corporis velocitas in
" iisdem locis.

" Sit

* Edit. Prin. 1687, pag. 260.

" Sit AK planum illud plano schematis perpendiculare;
" ACK linea curva; C corpus in ipsa motum; et FCf recta
" ipsam tangens in C. Fingatur autem corpus C nunc pro-
" gredi ab A ad K per lineam illam ACK, nunc vero regredi
" per eandem lineam; et in progressu impediri a medio, in
" regressu æque promoveri; sic ut in iisdem locis eadem
" semper sit corporis progredientis et regredientis velocitas.
" Æqualibus autem temporibus describat corpus progre-
" diens arcum quam minimum CG, et corpus regrediens ar-
" cum Cg; et sint CH, Ch longitudines æquales rectilineæ,
" quas corpora de loco C exeuntia, his temporibus, absque
" medii et gravitatis actionibus describerent: et a punctis
" C, G, g ad planum horizontale AK demittantur perpendi-
" cula CB, GD, gd, quorum Gd ac gd tangenti occurrant in
" F et f. Per medii resistentiam fit ut corpus progrediens,
" vice longitudinem CH describat solummodo longitudinem
" CF; et per vim gravitatis transfertur corpus de F in G:
" adeoque lineola HF vi resistentiæ et lineola FC vi gravi-
" tatis simul generantur. Proinde (per Lem. 10. Lib. I.)
" lineola FG est ut vis gravitates et quadratum temporis
" conjunctim, adoque (ob datam gravitatem) ut quadratum
" temporis et lineola HF ut resistentia et quadratum tempo-
" ris, hoc est ut resistentia et lineola FG. Et inde resistentia
" fit ut HF directe et FG inverse, sive ut $\frac{\mathrm{HF}}{\mathrm{FG}}$. Hæc ita se ha-
" bent in lineolis nascentibus. Nam in lineolis finitæ mag-
" nitudinis hæ rationes non sunt accuratæ.

" Et simili argumento est fg ut quadratum temporis,
" adeoque ob æqualia tempora æquatur ipsi FG; et impul-

" sus

" sus quo corpus regrediens urgetur est ut $\frac{hf}{fg}$. Sed impulsus
" corporis regredientis et resistentia progredientis ipso motus
" initio æquantur, adeoque et ipsis proportionales $\frac{hf}{fg}$ et $\frac{HF}{FG}$
" æquantur; et propterea ob æquales fg et FG, æquantur
" etiam hf et HF, suntque adeo CF, CH (vel Ch) et Cf in
" progressione Arithmetica, et inde HF semidifferentia est
" ipsarum Cf et CF; et resistentia quæ supra fuit ut $\frac{HF}{FG}$, est
" ut $\frac{Cf-CF}{FG}$.† * * * * *
" * * * * * * *

" Cor. 1. * * * * *
" * * * * * * *

" Erit enim fC ad kC ut $\sqrt{fg}$ seu $\sqrt{FG}$ ad $\sqrt{kl}$, et divisim
" fh ad kC, id est Cf—CF ad CF ut $\sqrt{FG}-\sqrt{kl}$ ad $\sqrt{kl}$;
" hoc est (si ducatur terminus uterque in $\sqrt{FG}+\sqrt{kl}$) ut
" FG—kl ad $kl+\sqrt{FG \times kl}$ sive ad FG+kl. Nam ratio prima
" nascentium $kl+\sqrt{FG \times kl}$ et FG+kl est æqualitatis. *
" * * * * * * *

" Cor. 2. Unde cum 2HF et Cf—CF æquentur, et FG
" et kl (ob rationem æqualitis) componant 2FG; erit 2HF
" ad FC ut FG—kl ad 2FG; et inde HF ad FG, hoc est
" resistentia ad gravitatem, ut rectangulum CF in FG—kl
" ad 4FG quad. * * * * *

" Cor. 3. Et hinc si curva linea definiatur per relationem
" inter basem seu abscissam AB et ordinatam applicatam

 " BC;

† The remainder of the solution only respects the law of variation of the density, and is therefore omitted; as well as the parts of the Corollaries which have not a reference to the general proportion of resistance to gravity.

" BC; (ut mos est) et valor ordinatim applicatæ resolvatur
" in seriem convergentem: Problema per primos serei termi-
" nes expedite solveretur: * * *
" * * * * * * *
" * * * Si designetur series universaliter
" his terminis $\mp Qo - Ro^2 - So^3$ &c. erit CF æqualis $\sqrt{o^2 + Qo^2}$
" * * FG—kl æqualis $2So^3$. * * Deducendo
" igitur Problema unumquodque ad seriem convergentem,
" et hic pro Q, R, S scribendo terminos serei ipsis respon-
" dentes; deinde etiam ponendo resistentiam medii in loco
" quovis C esse ad gravitatem ut $S\sqrt{1+Q^2}$ ad $2R^2$ *
" solvetur problema. * * *

Now with respect to this solution it may be remarked, that in Cor. 1. it is stated that

$$fC : kC :: \sqrt{fg} \text{ seu } \sqrt{FG} : \sqrt{kl} \text{ et}$$

divisim $fk : kC$ id est $Cf - CF : CF :: \sqrt{FG} - \sqrt{kl} : \sqrt{kl}$ But although ultimò $\sqrt{fg} : \sqrt{FG}$ is a ratio of equality, it does not follow that ultimò $\sqrt{FG} - \sqrt{kl} : \sqrt{fg} - \sqrt{kl}$ is a ratio of equality. It is easy to see, that ultimò $fg : kl$ is a ratio of equality; and, therefore, it by no means necessarily follows, from the method of prime and ultimate ratios, that if ultimò $\sqrt{fg} : \sqrt{FG}$ is a ratio of equality, that ultimò $\sqrt{FG} - \sqrt{kl} : \sqrt{fg} - \sqrt{kl}$ is also a ratio of equality. Had ultimò $fg : kl$ not been a ratio of equality, then $\sqrt{FG} - \sqrt{kl} : \sqrt{fg} - \sqrt{kl}$ must necessarily have been a ratio of equality. Because ultimò $\sqrt{fg}$, $\sqrt{FG}$, $\sqrt{kl}$, are all equal, we may represent them by $ao + bo^2 +$ &c. $ao + {'b}o^2 +$ &c. $ao + {''b}o^2 +$ &c. where o may be diminished indefinitely. Then ultimò

ultimò $\sqrt{fg}-\sqrt{kl} : \sqrt{FG}-\sqrt{kl} :: b-b'' : b'-b''$. Now there is nothing in the Newtonian solution by which it can be shewn that this ratio is a ratio of equality. It may or may not be so; and, therefore, from this step, the Newtonian investigation ceases to be supported by demonstration.

Let us suppose that ultimò

$$Cf-CF : CF :: m\ (\sqrt{FG}-\sqrt{kl}) : \sqrt{kl}$$

If m be found to be unity, then Newton's proportion is accurate, otherwise not.

Proceeding with this corollary, in the manner of Newton, we have

$$Cf-CF : CF :: m\ (FG-kl) : CG+hl.$$

Also for Cor. 2.

$$2HF : CF :: m\ (FG-kl) : 2FG$$
$$CF : 2FG :: CF : 2FG$$

Therefore

$$2HF : 2FG :: m\ (FG-kl)\ CF : 4FG^2$$

And hence Resist. : Grav. :: $mS\sqrt{1+Q^2} : 2R^2$

But according to the corrected Newtonian solution, as given in the second and third editions of the Principia, as given also by Lagrange, and as is likewise shewn at the end of this paper,

$$\text{Resist. : Grav. :: } 3S\sqrt{1+Q^2} : 4R^2$$

Hence $m = \frac{3}{2}$, and therefore ultimò

$$\sqrt{fg}-\sqrt{kl} : \sqrt{FG}-\sqrt{kl} :: 3 : 2,$$

instead of the ratio of equality *assumed* by Newton.

The ultimate ratio of $\sqrt{fg}-\sqrt{kl} : \sqrt{FG}-\sqrt{kl}$, or the ratio $m : 1$ may also be fluxionally investigated as follows:

 Let

Let x=AB, y=BC, x'=BD, and Resist. : Grav. :: $r : g$,

Then $\text{FG} = \frac{\ddot{y}}{2\dot{x}^2}x'^2 + \frac{\overset{3}{\dot{y}}}{6\dot{x}^3}x'^3 + \&c.$

$$kl = \frac{\ddot{y}}{2\dot{x}^2}x'^2 - \frac{\overset{3}{\dot{y}}}{6\dot{x}^3}x'^3 + \&c.$$

Or $\sqrt{\text{FG}} = \left(\frac{\ddot{y}}{2\dot{x}^2}\right)^{\frac{1}{2}}x' + \left(\frac{\ddot{y}}{2\dot{x}^2}\right)^{-\frac{1}{2}}\frac{\overset{3}{\dot{y}}}{12\dot{x}^3}x'^2 + \&c.$

$$\sqrt{kl} = \left(\frac{\ddot{y}}{2\dot{x}^2}\right)^{\frac{1}{2}}x' - \left(\frac{\ddot{y}}{2\dot{x}^2}\right)^{-\frac{1}{2}}\frac{\overset{3}{\dot{y}}}{12\dot{x}^3}x'^2 + \&c.$$

Hence $\sqrt{\text{FG}} - \sqrt{kl} = \left(\frac{\ddot{y}}{2\dot{x}^2}\right)^{-\frac{1}{2}}\frac{\overset{3}{\dot{y}}}{6\dot{x}^3}x'^2 + \&c.$

Now ultimò FG : FH :: $g : r$,

Therefore let $\text{FH} = \frac{r}{g}\text{FG} + a\text{FG}^2 + \&c.$

And let $fh = \frac{r}{g}fg + a'fg^2 + \&c.$

Then $\text{C}f = \text{CF} + \frac{r}{g}(\text{FG}+fg) + a\text{FG}^2 + a'fg^2 + \&c.$

Therefore $\text{C}m = x' + \frac{\dot{x}r}{\dot{z}g}(\text{FG}+fg) + \frac{\dot{x}}{\dot{z}}(a\text{FG}^2 + a'fg^2) + \&c.$

But $\sqrt{fg} = \left(\frac{\ddot{y}}{2\dot{x}^2}\right)^{\frac{1}{2}}\text{C}m - \left(\frac{\ddot{y}}{2\dot{x}^2}\right)^{-\frac{1}{2}}\frac{\overset{3}{\dot{y}}}{12\dot{x}^3}\text{C}m^2$ &c.

Therefore $\sqrt{fg} = \left(\frac{\ddot{y}}{2\dot{x}^2}\right)^{\frac{1}{2}}\left(x' + \frac{\dot{x}r}{\dot{z}g}(\text{FG}+fg)\right) - \left(\frac{\ddot{y}}{2\dot{x}^2}\right)^{-\frac{1}{2}}\frac{\overset{3}{\dot{y}}}{12\dot{x}^3}x'^2 +$ &c.

Consequently $\sqrt{fg} - \sqrt{kl} = \left(\frac{\ddot{y}}{2\dot{x}^2}\right)^{\frac{1}{2}}\frac{\dot{x}r}{\dot{z}g}\frac{\ddot{y}}{\dot{x}^2}x'^2 + \&c. = \left(\text{because } \frac{r}{g} = \frac{\dot{z}\overset{3}{\dot{y}}}{2\ddot{y}^2}\right)\frac{\overset{3}{\dot{y}}}{2\sqrt{2}\,\ddot{y}^{\frac{1}{2}}\dot{x}^2}x'^2 + \&c.$ Hence ultimò

$$\sqrt{fg} - \sqrt{kl} : \sqrt{\text{FG}} - \sqrt{kl} :: \frac{1}{2\sqrt{2}} : \frac{\sqrt{2}}{6} :: 3 : 2.$$

To

To obtain this conclusion we have used the true values of $\frac{r}{g}$ as found by the method of series, and also by the method of fluxions. The value of $\frac{r}{g}$ enables us readily to compute the value of $\sqrt{fg}-\sqrt{kl}$; and it does not appear that there is any other convenient method of obtaining that value. Hence the Newtonian method of solution, proceeding with $\sqrt{fg}-\sqrt{kl}$, instead of $\sqrt{FG}-\sqrt{kl}$, will require a process too complicated to be pursued with convenience: and it is highly probable, that Newton, revising his solution, discovered the true source of error, and thence was induced to abandon it entirely.

Of this solution of Newton Lagrange observes as follows: "Voici la première solution de Newton reduite en analyse:" (Œuvres de Jean Bernouilli, Tom. I. p. 481.) "Le mobile
" étant parvenu à un point quelconque de la courbe, sans la
" résistance et la gravité il décrirait dans un temps donné
" très-petit, une partie très-petite de la tangente que nous
" désignerons par α; soit γ le petit espace que la gravité
" ferait décrire dans le même temps perpendiculairement à
" l'horizon, et ρ le petit espace dont la résistance diminue
" l'espace α parcouru sur la tangente, il est clair que le
" rapport de ρ à γ sera celui de la résistance à la gravité.
" Ainsi le corps, dans le temps qu'il auroit parcouru sur la
" tangente l'espace $\alpha-\rho$, sera descendu vertualement de la
" quantité γ; par consequent γ sera la flèche de l'arc $\alpha-\rho$
" Maintenant, si on considèré le corps comme partant du
" même

* Theo. des Fonct. analyt. p. 244.

" même point et rebroussant chemin par décrire en sens " contraire le même arc de courbe qu'il a parcouru, il faudra " regarder le résistance comme négative, et par consequent " comme une force qui accélère le mouvement au lieu de la " retarder. Ainsi, le corps décrira, dans le même temps " très-petit, l'espace $\alpha+\rho$ sur le même tangente dans une " direction contraire, et descendra en même temps verticale- " ment de l'espace γ, en vertu de la gravité. Par consequent " γ sera la flèche de l'arc $\alpha+\rho$, pris de l'autre côté du part " de la courbe dont il s'agit. Or, les flèches étant pour les " arcs infiniment petits, comme les carrés des arcs, ou des " tangentes, la flèche de l'arc $\alpha-\rho$, pris du même côté que " l'arc $\alpha+\rho$, sera $\gamma\left(\frac{\alpha-\rho}{\alpha+\rho}\right)^2$; donc, la différence des flèches " pour les arcs égaux $\alpha-\rho$, pris de part et d'autre du point " donné de la courbe, sera $\gamma\left(1-\frac{(\alpha-\rho)^2}{(\alpha+\rho)^2}\right) = \frac{4\alpha\gamma\rho}{(\alpha+\rho)^2}$: nommons " cette difference δ, on aura $\frac{4\alpha\gamma\rho}{(\alpha+\rho)^2} = \delta$, et $\frac{\rho}{\gamma} = \frac{\delta(\alpha+\rho)^2}{4\gamma^2\alpha} = \frac{\delta\alpha}{4\gamma^2}$, " à cause que la petite ligne ρ, parcourue d'un mouvement " uniformément accéléré, est infiniment plus petite que la " ligne α parcourue dans le même temps d'un mouvement " uniforme. Tel est raisonnement de Newton, présenté de " la manière la plus claire; et le résultat que nous venons " de trouver, s'accorde avec celui du corollaire 2. du pro- " blemé, où il est visible que les lignes CF et FG sont ce " que nous avons nommé α et γ, et que la différence " FG—Kl est ce que nous avons nommé δ."

He then proceeds to investigate the value of $\frac{\delta a}{4\gamma^2}$, and deduces the same conclusion as Newton.

Now

Now it does not immediately appear, that the result from this solution should be the same as the erroneous result of Newton; but a little consideration will make it apparent. fg is made to disappear in both solutions, by the substitution of FG, and therefore a common error might be expected in each solution. This is proved as follows: let γ' represent the subtense of the arc $\alpha+\rho$. Then FG—kl or δ will be $\gamma-\gamma'\left(\frac{\alpha-\rho}{\alpha+\rho}\right)^2$ or $\frac{\alpha^2(\gamma-\gamma')+2\alpha\rho(\gamma+\gamma')+(\gamma-\gamma')\rho^2}{(\alpha+\rho)^2}$. This latter quantity, therefore, cannot become ultimately $\frac{4\alpha\gamma\rho}{(\alpha+\rho)^2}$, unless $\alpha(\gamma-\gamma')$ vanishes in respect to $4\gamma\rho$. Now it may be readily shewn, in a manner similar to that in which the limiting ratio of $\sqrt{fg}-\sqrt{kl} : \sqrt{FG}-\sqrt{kl}$ was obtained, that ultimò $\alpha(\gamma-\gamma')= -\frac{4}{3}\gamma\rho$. It is easy to see that $\rho^2(\gamma-\gamma')$ vanishes in respect to $\alpha^2(\gamma-\gamma')$. But that $\alpha(\gamma-\gamma')$ vanishes in respect $4\gamma\rho$ is only an assumption.

The other solution of Lagrange, in which the same conclusion is deduced, is the following.

" La solution de Newton peut être rendue plus simple et
" plus directe de la manière suivante. En nommant u la
" vîtesse dans un point quelconque de la courbe, $u\theta$ est
" l'espace que le corps parcourait sur la tangente dans le
" temps θ, en faisant abstraction de la gravité et de la
" résistance. Nommont g la force absolue de la gravité,
" et r celle de la resistance, $\frac{g\theta^2}{2}$ et $\frac{r\theta^2}{2}$ seront les espaces par-
" courus, en vertu de ces forces, dans le même temps θ;
" ainsi le corps aura parcouru, suivant la tangente, la ligne
" $u\theta-\frac{r\theta^2}{2}$, et suivant l'ordonnée y, la ligne $\frac{g\theta^2}{2}$, dans une
" direction contraire à celle suivant laquelle cette coor-
donnée

" donnée croît. Soit A l'ángle de la tangente avec l'axe des " x, il en résultera, suivant la direction de l'axe des x, " l'espace $(u\theta - \frac{r\theta^2}{2})\cos A$, et suivant la direction de l'axe " des y, l'espace $(u\theta - \frac{r\theta^2}{2})\sin A - \frac{g\theta^2}{2}$. Or, y etant onction " de x, supposons, avec Newton, que x devenant $x+o$, " y devienne $y + Qo - Ro^2 - So^3 -$ &c.; il faudra donc, qu'en " faisant $o = (u\theta - \frac{r\theta^2}{2})\cos A$, on ait $Qo - Ro^2 - So^3$, &c. $=$ " $(u\theta - \frac{r\theta^2}{2})\sin A - \frac{g\theta^2}{2}$, quelle que soit la valeur de θ, " qu'on suppose très petite.

" Substituons, dans la seconde équation, la valeur de " o donnée par la première, et ordonnant les termes par " rapport aux puissances de θ, on aura $\theta Qu \cos A - $ " $(\frac{Qr}{2}\cos A + Ru^2 \cos A^2)\theta^2 - (Rur \cos A^2 + Su^3 \cos A^3)\theta^3 +$ &c. $=$ " $\theta u \sin A - (\frac{r \sin A}{2} + \frac{g}{2})\theta^2$. Comparont terme à terme, on a " $Qu \cos A = u \sin A$, $Qr \cos A + 2Ru^2 \cos A^2 = r \sin A + g$, " $Rur \cos A^2 + Su^3 \cos A^3 = o$, &c. La première équation " donne tang. $A = Q$; substituont cette valuer dans le " seconde, on a $2Ru^2 \cos A^2 = g$, d'où l'on tire $u^2 = \frac{g}{2R \cos A^2} =$ " $\frac{g(1+Q^2)}{2R}$; le troisième donne $r = \frac{Su^2 \cos A}{R}$, substituont pour " u^2 et pour $\cos A$ leurs valeurs, on aura $r = \frac{gS\sqrt{1+Q^2}}{2R^2}$, et de-là " $\frac{r}{g} = \frac{S\sqrt{1+Q^2}}{2R^2}$, rapport de la résistance a la gravité, comme " Newton l'avait trouvé. En effet, il est facile de voir que " cette analyse n'est, au fond, que celle de Newton debar- " rassée de la considération des deux mouvemens en sens " contraire, et réduite à la forme la plus simple; mais elle a, " de plus, l'avantage de faire connaître facilement la source " de l'erreur, et de donner le moyen d'y remedier.

" Car, pour peu qu'on examine le calcul que nous venons " de

" de faire, en doit voir que, puisque les valeurs de o et de " $Qo—Ro^2—So^3$-- &c. sont exprimées en séries qui pro- " cédent suivant les puissances de θ, il n'est pas permis de " pousser l'approximation au-delà de cette même puissance " dans l'équation résultant de l'élimination de o: d'où il suit " que le terme que contient θ^3 dans cette équation, dont " nécessairement être incomplet; et puisque c'est de ce " même terme que dépend le rapport cherché de $\frac{r}{g}$, on en " doit conclure que le valeur trouvée de ce rapport est " inexacte."

Notwithstanding the remarks of the ingenious author, it is not very clear, that the error of the result in this solution must necessarily be the same as in that of Newton, if the error of the Newtonian solution have been rightly pointed out. These solutions have nothing in common; and, therefore, as they give the same result, the error in each must flow from a common source. In this solution of Lagrange, he computes the increments of the ordinate and abscissa in the time θ, by supposing the resistance to act, during that time, in the direction of the tangent; and thus the deviation from the tangent in the time θ is expressed by $\frac{g\theta^2}{2}$, depending only on the time and force of gravity. In the Newtonian solution, fg and FG, the deviations from the tangent in equal times, are taken accurately equal, and therefore made to depend on the force of gravity only. Hence, a common source of error; and these solutions, so entirely different in their progress, might be expected to produce the same result.

M. Lagrange conclndes his observations on this problem,

 by

by shewing how his solution, given above, may be made to produce a true result. He obtains the coefficients of θ^3 in the increments of the abscissa and ordinate, from those of o, and θ^2, by an exceedingly ingenious process; having so deduced o, and $Qo - Ro^2 - So^3$, &c. complete as far as θ^3, he exterminates o as before, and by a comparison of the coefficients (now correct) of θ^3 he obtains the relation of gravity and resistance.

The ingenuity exhibited in the corrected solution is very great, but it must be confessed that the solution is rather prolix. The author concludes his corrected solution with these words, which may serve as an excuse for adding that which concludes this paper.

" Comme Newton n'est parvenu à ce dernier résult, qu'en " suivant une marche analogue à celle du calcul differentiel, " nous avons cru qu'il n'etait pas inutile de faire voir " comment la methóde des séries pouvait y conduire, et " qu'on nous saurait gré d'éclaircir, en même temps un " point d'analyse sur lequel les plus grand géomètres " s'etoient trompés, et qui peut intéresser l'histoire de la " naissance des nouveaux calculs.

THEOREM.

THEOREM.* If $BD=o$, and $IG=Qo+Ro^2+So^3+$ &c. Then Resistance (r) : Gravity (g) :: $3S\sqrt{1+Q^2}$: $4R^2$.

Dem. Let t=time of describing CG, then the velocity at C in direction CI=limit of $\frac{CI}{t}$=limit of $\frac{CI}{\sqrt{FG}}\times\sqrt{\frac{1}{2}g}$. But Resist. in direction CI = lim. $\frac{\text{dec. vel.}}{t}$ = lim. $\frac{\text{dec. lim.}\frac{CI}{\sqrt{FG}}\times\frac{1}{2}g}{\sqrt{FG}}$. Now $FG=IG-FI=Ro^2+So^3+$ &c. Therefore limit $\frac{CI}{\sqrt{FG}}=\frac{1}{\sqrt{R}}$ and dec. $\frac{1}{\sqrt{R}}=\dagger\frac{3So}{2R^{\frac{3}{2}}}$ &c. therefore lim. $\frac{\text{dec.}\frac{1}{\sqrt{R}}}{\sqrt{FG}}=\frac{3S}{2R^2}$. Hence resist. in direct. CI $=\frac{3Sg}{4R^2}$ and consequently whole resistance $=\frac{3Sg}{4R^2}\times$ lim. $\frac{CF}{CI}=\frac{3Sg}{4R^2}\sqrt{1+Q^2}$ and $r:g::3S\sqrt{1+Q^2}:4R^2$. Q. E. D.

* Vid. Fig.

† When CI or o becomes $o+\acute{o}$, IG becomes $Q(o+\acute{o})+R(o+\acute{o})^2+S(o+\acute{o})^3+$ &c. =

$$\left\{\begin{array}{l}(Q+2R\acute{o}+3S\acute{o}^2)o\\+(R+3S\acute{o}+6T\acute{o}^2)o^2\\\text{\&c. \&c.}\end{array}\right.\qquad\begin{array}{l}\text{Hence when AB is increased by } o,\\\text{Incr. of } Q=2Ro+3So^2+\text{ \&c.}\\\text{Incr. of } R=3So+6To^3\text{ \&c.}\\\text{\&c. \&c.}\end{array}$$

Therefore dec. $\frac{1}{\sqrt{R}}=\frac{1}{\sqrt{R}}-\frac{1}{\sqrt{R+3So+\text{\&c.}}}=\frac{1}{\sqrt{R}}\times\frac{3So}{2R}$ &c.